CHAPTER 12
QUANTIFICATION THEORY
PROOFS: MAIN

THE RULES

In this chapter we will introduce eight new rules. These rules, when used in conjunction with those presented in our earlier chapter on Proofs, will permit us to construct proofs in quantification theory.

Henceforth, any argument whose conclusion follows from its premises using both the new and old rules will be called *quantificationally valid*; while any argument whose conclusion follows from its premises using only the earlier rules, will be called *truth-functionally valid*. So, every truth-functionally valid argument will also be quantificationally valid, but not every quantificationally valid argument will also be a truth-functionally valid one.

Two of our eight new rules -- Assumption and Reiteration -- might best be labeled special rules. We will discuss them first although the reason for having them will not be entirely clear until we have explored some of the remaining rules. Our third rule -- Conditional Proof -- could have been presented in our earlier chapter on proofs. It functions as a rule of inference. Rules four through seven, are all Rules of Inference that deal with quantifiers. The fourth rule -- Universal Quantifier Generalization -- tells us how to introduce a universally quantified formula. The fifth rule -- Universal Quantifier Instantiation -- tells us how to use a universally quantified formula. The sixth rule -- Existential Quantifier Generalization -- tells us how to create an existentially quantified formula. And the seventh -- Existential Quantifier Instantiation -- tells us how to use an existentially quantified formula. Finally, our eighth rule -- Quantifier Equivalence -- is a Replacement Rule for exchanging quantifiers.

ASSUMPTION

We may use the rule Assumption, abbreviated Ass, any time, and we can assume any formula we wish. When we use this rule, however, it blocks off all future formulas until the assumption is discharged. The proof is only completed when the conclusion is obtained after all assumptions have been discharged. (On the computer disks, in both the practice exercises and the examination, no points are given for using this rule.) We will see how to discharge assumptions shortly.

1	**(P>Q)**	**/(R>((S=-T)>(P>Q)))**	**Premise**
2	**┣ R**		**Ass**
3	**┃ ┣ (S=-T)**		**Ass**

To use the rule Assumption, we type "Ass" when we are asked which rule we want to use. Then, we simply type in the formula we want to assume.

REITERATION

The rule Reiteration, abbreviated Reit, permits us to repeat a formula into an assumed block. Once an assumption has been discharged, however, no formula listed under that assumption can be reiterated elsewhere. To use the rule, we type "Reit" when we are asked which rule we want to use, and then we type the formula we want to reiterate. (As with Assumption, no credit is given for using Reit.) Study the examples below.

CORRECT			*INCORRECT*			
1	(P>Q)	Premise	1	Q	Premise	
2	│ P	Ass	2	│ P	Ass	
3	│ (P>Q)	1, Reit	3	│ (PvR)	2, Add	
4	│ Q	2, 3, MP	4	P	2, Reit *	ILLEGAL
			5	│ R	Ass	
			6	│ (PvR)	3, Reit *	ILLEGAL

In the example on the right, on line 3 we discharged the assumption on line 2, so we cannot reiterate the formula (PvR) on line 6.

CONDITIONAL PROOF

Conditional Proof, abbreviated CP, is one of the two rules in our system that permits us to discharge an assumption. To use this rule, we must already have made an assumption and obtained a formula under that assumption. CP permits us to end that assumption and build a horseshoe claim whose left side is the assumption we made, and whose right side is the formula we obtained under that assumption. After the assumption has been discharged, neither it nor any formula obtained under it, can be reiterated or otherwise used. Examine the examples below.

EXAMPLES

1	((P>R)>S)	Premise	1	((R>((P=T)>R))>S)	Premise
2	R	Premise	2	│ R	Ass
3	│ P	Ass	3	│ │ (P=T)	Ass
4	│ R	2, Reit	4	│ │ R	2, Reit
5	(P>R)	3-4, CP	5	│ ((P=T)>R)	3-4, CP
6	S	1, 5, MP	6	(R>((P=T)>R))	2-5, CP
			7	S	1, 6, MP

UNIVERSAL QUANTIFIER GENERALIZATION

So far, although the rules we have added to our old system may make some of our proofs less lengthy than they otherwise would be, they will not allow us to derive the conclusions of any valid arguments from their premises that we could not have derived before. However, with Universal Quantifier Generalization, abbreviated UG, everything changes.

The rule UG is a building rule. It permits us to create a universally quantified formula. To create such a formula, all we need to do is go through the following process:

1. Select a formula that has already occurred in the proof and that we want to use UG on.
2. Select a constant (viz., a-u) that has not occurred in any premise or in any un-discharged assumption. (Normally, this constant will occur in the formula we have selected, but it need not have.)
3. Select a variable (viz., w-z).
4. Replace every occurrence of the constant selected in the formula we have decided to use UG on with the variable we have chosen.
5. Precede the result with (, followed by the chosen variable, followed by).

Thus, suppose Paba is the formula we want to use UG on, suppose the constant we have selected is "a," and that this constant doesn't occur in any premise or un-discharged assumption. Suppose, further, that the variable we have chosen is "x." Using UG on Paba, we will get (x)Pxbx. Study the examples below.

CORRECT			INCORRECT		
1	((x)(Px>Px)>Ra)	Premise	1	((x)(Px>Px)>Ra)	Premise
2	│Pb	Ass	2	│Pa	Ass
3	│Pb	2, Reit	3	│Pa	2, Reit
4	(Pb>Pb)	2-3, CP	4	(Pa>Pa)	2-3, CP
5	(x)(Px>Px)	4, UG	5	(x)(Px>Px)	4, UI * ILLEGAL
6	Ra	1, 5, MP	6	Ra	1, 5, MP

Although these proofs are very similar, the one on the right makes the mistake of trying to generalize on a constant (viz., "a") that occurs in the premise on line 1. Why do we have this restriction? Because, if we didn't, we could go from a premise that said, for example, Albert is happy, to the claim that everyone is happy. In the example on the left, it didn't matter that "b" was selected. We could have chosen any constant. So our reasoning is true of everyone.

UNIVERSAL QUANTIFIER INSTANTIATION

This rule, abbreviated UI, is quite simple. To use it, all we need to have is a universally quantified formula occurring on an earlier line. We simply delete the quantifier, and replace every occurrence of the variable that occurred in that quantifier, with a constant of our choice. (Normally, the constant we choose will have occurred before, but this is not necessary.) Consider the example below.

1	(x)(Px>Qx)		Premise
2	(x)(-Rx>-Qx)	/ (x)(Px>Rx)	Premise
3	(Pa>Qa)		1, UI
4	(-Ra>-Qa)		2, UI
5	(Qa>Ra)		4, Trans
6	(Pa>Ra)		3, 5, HS
7	(x)(Px>Rx)		6, UG

EXISTENTIAL QUANTIFIER GENERALIZATION

The rule Existential Quantifier Generalization, abbreviated EG, is even easier than UI. It builds an existentially quantified formula. To use it, we simply select an earlier line in the proof, replace any number of occurrences of a constant of our choice with whatever variable we want, and then precede the result with (E, followed by that variable, and then). Thus, using EG on Paba, we can obtain, for example, (Ex)Pxbx, or (Ex)Pxba, or (Ey)Pyba, or (Ez)Pabz, or even (Ex)Paba.

The idea of this rule is that since a particular individual has a certain property, it follows that someone or something has that property. Study the following proof.

1	((Ex)(Ey)Pxy>(z)Qz)		Premise
2	Paa	/ Qa	Premise
3	(Ey)Pay		2, EG
4	(Ex)(Ey)Pxy		3, EG
5	(z)Qz		1, 4, MP
6	Qa		5, UI

Note the way in which (Ex)(Ey)Pxy gets built. We build it from Paa by adding existential quantifiers from right to left. We could, of course, have chosen variables other than x and y, but we selected the ones we did so we could use MP on line 1.

EXISTENTIAL QUANTIFIER INSTANTIATION

This is the toughest rule in the whole system. Like CP, it is a rule that permits us to discharge an assumption, and like UG, it contains restrictions. However, the restrictions here are enough to drive one crazy.

To use the rule Existential Quantifier Instantiation, abbreviated EI, we must have an existentially quantified formula already listed. (It is the line number of this formula which we will enter when we are asked what line we want to use the rule on.) Moreover, we must have made an assumption that is an instance of the existentially quantified formula we are going to use EI on.

To obtain an instance of the existentially quantified formula, delete the quantifier and replace every occurrence of the variable that was in that quantifier with a constant. Make sure the constant selected does not occur in any premise or un-discharged assumption, and that it does not occur in the existentially quantified formula, or we will be violating one of the restrictions on using the rule.

Once we have derived a formula under the assumption we have made, we may discharge that assumption and enter the same formula we obtained under that assumption left one line. However, there is a restriction here. The formula we are moving left must not contain any occurrences of the constant that we selected in the assumption we are discharging. (This is the formula we are obtaining when we use EI.)

Let's look at some examples. We'll see how not to do it first. Then we will see how it should be done, and why.

INCORRECT USES OF EI

1	(Ex)Px		Premise
2	(x)(Px>Qx)		Premise
3	((Ex)Qx>Ra)	/ (Ex)Rx	Premise
4	| Pa		Ass
5	| (Pa>Qa)		2, UI
6	| Qa		4, 5, MP
7	| (Ex)Qx		6, EG
8	(Ex)Qx	ILLEGAL	1, 4-7, EI *

The use of EI here is illegal because the constant selected on line 4 occurs in the premise on line 3.

1	(Ex)Px		Premise
2	(x)(Px>Qx)		Premise
3	((Ex)Qx>Ra)		Premise
4	| Pb		Ass
5	| (Pb>Qb)		2, UI
6	| Qb		4, 5, MP
7	Qb	ILLEGAL	1, 4-6, EI *

The use of EI here is illegal because the constant selected on line 4 occurs in the formula you are pulling out of the assumed block.

Now let's redo this problem. However, let's do it correctly this time.

AN EXAMPLE OF A CORRECT USE OF EI

1	(Ex)Px		Premise
2	(x)(Px>Qx)		Premise
3	((Ex)Qx>Ra)	/ (Ex)Rx	Premise
4	| Pb		Ass
5	| (Pb>Qb)		2, UI
6	| Qb		4, 5, MP
7	| (Ex)Qx		6, EG
8	(Ex)Qx		1, 4-7, EI
9	Ra		3, 8, MP
10	(Ex)Rx		9, EG

Note that the line number we are using EI on is 1. It is this line number that we cite, when we are asked, "Which earlier line do you want to use this rule on?" The block starting with line 4 is also cited. The assumed formula on line 4 is an instance of the existentially quantified formula, and the constant selected, "b," doesn't occur in line 1, or in any premise or un-discharged assumption, or in line 8.

Why all the restrictions on EI? They are, of course, designed to prevent us from deriving the conclusions of invalid arguments from their premises. Suppose, for example, that when you are creating the assumed instance of the existentially quantified formula, we permitted you to select a constant that had already occurred in a premise, or an un-discharged assumption, or in the existentially quantified formula. From a claim like, "Someone is shorter than Albert," you could then easily derive the claim that "Someone is shorter than he himself is." The proof might proceed as follows:

1	(Ex)Sxa	Premise
2	| Saa	Ass
3	| (Ex)Sxx	2, EG
4	(Ex)Sxx	1, 2-3, EI * ILLEGAL

While we can't prevent you from assuming Saa on line 2, because the rule Assumption permits you to assume anything, our restrictions do prohibit the use of EI on line 4.

Or, suppose we allowed you to pull a formula out of the assumed block that contained the constant selected when you created the assumed instance of the existentially quantified formula. Then, from "Someone is happy," you could derive the claim that "Albert is happy." The proof would develop as follows:

1	(Ex)Hx	Premise
2	| Ha	Ass
3	| Ha	2, Reit
4	Ha	1, 2-3, EI * ILLEGAL

Unfortunately, it is all the restrictions that make EI such a difficult rule. Before you use the rule, please check to see that you haven't violated them.

QUANTIFIER EQUIVALENCE

As the name suggests, Quantifier Equivalence, abbreviated QEQ, is an equivalence or replacement rule, and so we can use it on a part of a formula. It tells us that a tilde followed by an existential quantifier is the same thing as a universal quantifier followed by a tilde; or, in ordinary English, it tells us that "It is not true that something has such-and-such" is the same thing as "Everything lacks such-and-such." Alternately, it tells us that a tilde followed by a universal quantifier is the same thing as an existential quantifier followed by a tilde. In other words, it tells us, "It is not true that all things have such-and-such" is the same thing as, "Some things lack such-and-such." Symbolically:

-(Ex)Px = (x)-Px and -(x)Px = (Ex)-Px

The following example uses QEQ:

1	-(Ex)-(-(y)Pxy>Qx)		Premise
2	(x)(Rx>(Ey)-Pxy)	/ (x)(Rx>(Ey)Qy)	Premise
3	Ra		Ass
4	--(x)(-(y)Pxy>Qx)		1, QEQ
5	(x)(-(y)Pxy>Qx)		4, DN
6	(-(y)Pay>Qa)		5, UI
7	(Ra>(Ey)-Pay)		2, UI
8	((Ex)-Pay>Qa)		6, QEQ
9	(Ra>Qa)		7, 8, HS
10	Qa		3, 9, MP
11	(Ey)Qy		10, EG
12	(Ra>(Ey)Qy		3-11, CP
13	(x)(Rx>(Ey)Qy)		12, UG

STRATEGIES

The following ideas may help some in your construction of proofs:

1. Look first at the premises and see if any of them are either negated existentially quantified formulas, or negated universally quantified formulas. If so, use QEQ to push the tilde in. Keep using this rule until the tilde is to the right of all the quantifiers. Do the same for the conclusion.

2. Now carefully examine all the premises you didn't use QEQ on, and all the claims you got when you used QEQ. If any of these claims are existentially quantified formulas, you need to assume an instance of them, and work toward using EI. If any contain two existential quantifiers in succession (e.g., (Ex)(Ey)Pxy), you'll need to make two assumptions. (In the example above, first assume (Ey)Pay, and then assume Pab.)

3. Now look at the conclusion. If it's a universally quantified formula, imagine it without the quantifier. At each point where the variable that was in that quantifier occurs you need to have the same constant, and that constant must not occur in the initial list of formulas.

4. If you are looking for a horseshoe claim, assume its left side and try to get its right side. Then use CP.

5. Now look back at your initial list of formulas. If any of them are universally quantified formulas, use UI on them. Select constants here that have occurred before.

6. At this point, try to use the rules and the strategies for them that we examined in the earlier chapter on Proofs.

7. Work from the bottom of the problem up, and then start at the top and work down. After you have made some moves, go back to where you left off at the bottom, and work up some more. Then try to make more moves from the top down.

8. If you can't solve the problem, put it away for a few minutes.

Let's see how these ideas might work in practice.

We know we have succeeded when we get the conclusion listed. So let's start by writing it down at the bottom of the page.

1	-(x)(y)(PxvQy)		Premise
2	(x)(-Px>(y)(Ray>Sx))	/ (x)(Rax>(Ey)Sy)	Premise
3	(Ex)-(y)(PxvQy)		1, QEQ
4	(Ex)(Ey)-(PxvQy)		3, QEQ

(x)(Rax>(Ey)Sy)

The formula on line 1 is a negated-universally quantified formula, so let's use QEQ to push the tilde to the right of the quantifiers.

1	-(x)(y)(PxvQy)	Premise
2	(x)(-Px>(y)(Ray>Sx)) / (x)(Rax>(Ey)Sy)	Premise
3	(Ex)-(y)(PxvQy)	1, QEQ
4	(Ex)(Ey)-(PxvQy)	3, QEQ
	(x)(Rax>(Ey)Sy)	

Now let's start working from the top of the problem down. Since the formula we have written on line 4 is an existentially quantified formula, we need to assume an instance of it. We must not use the constant "a," however, since it already occurs in the second premise. So let's use "b."

1	-(x)(y)(PxvQy)	Premise
2	(x)(-Px>(y)(Ray>Sx)) / (x)(Rax>(Ey)Sy)	Premise
3	(Ex)-(y)(PxvQy)	1, QEQ
4	(Ex)(Ey)-(PxvQy)	3, QEQ
5	| (Ey)-(PbvQy)	Ass
	(x)(Rax>(Ey)Sy)	

Since the assumption we just made is also an existentially quantified formula, we need to assume an instance of it. Here again however, we must select a new constant. Let's choose "c." Our problem will then look like this:

```
1  -(x)(y)(PxvQy)                              Premise
2  (x)(-Px>(y)(Ray>Sx))   /  (x)(Rax>(Ey)Sy)   Premise
3  (Ex)-(y)(PxvQy)                             1, QEQ
4  (Ex)(Ey)-(PxvQy)                            3, QEQ
5  | (Ey)-(PbvQy)                              Ass
6  | | -(PbvQc)                                Ass
   | |
   |
   (x)(Rax>(Ey)Sy)
```

The idea now is to get the conclusion under this assumption and then pull this conclusion over twice using EI. Let's work on the end of the proof some.

```
1  -(x)(y)(PxvQy)                              Premise
2  (x)(-Px>(y)(Ray>Sx))   /  (x)(Rax>(Ey)Sy)   Premise
3  (Ex)-(y)(PxvQy)                             1, QEQ
4  (Ex)(Ey)-(PxvQy)                            3, QEQ
5  | (Ey)-(PbvQy)                              Ass
6  | | -(PbvQc)                                Ass
   | |
   | (x)(Rax>(Ey)Sy)
   (x)(Rax>(Ey)Sy)
```

Our goal is obviously to get a universally quantified formula. To get it we need to obtain an instance of it where the constant we are generalizing on doesn't occur before. Let's assume this constant is "d." Then the formula we have to obtain is (Rad>(Ey)Sy).

```
1  -(x)(y)(PxvQy)                              Premise
2  (x)(-Px>(y)(Ray>Sx))   /  (x)(Rax>(Ey)Sy)   Premise
3  (Ex)-(y)(PxvQy)                             1, QEQ
4  (Ex)(Ey)-(PxvQy)                            3, QEQ
5  | (Ey)-(PbvQy)                              Ass
6  | | -(PbvQc)                                Ass
   | |
   | | (Rad>(Ey)Sy)
   | | (x)(Rax>(Ey)Sy)
   | (x)(Rax>(Ey)Sy)
   (x)(Rax>(Ey)Sy)
```

This formula is a horseshoe claim. We know we can get it by using CP if we assume its left side and obtain its right side under that assumption. So let's assume Rad, and try to get (Ey)Sy under this assumption.

We now need to get (Ey)Sy, and we are going to get this by using EG after we have obtained an

```
1  -(x)(y)(PxvQy)                                     Premise
2  (x)(-Px>(y)(Ray>Sx))   /  (x)(Rax>(Ey)Sy)           Premise
3  (Ex)-(y)(PxvQy)                                    1, QEQ
4  (Ex)(Ey)-(PxvQy)                                   3, QEQ
5  |(Ey)-(PbvQy)                                      Ass
6  ||-(PbvQc)                                         Ass
7  |||Rad                                             Ass
   |||
   |||(Ey)Sy
   ||(Rad>(Ey)Sy)
   ||(x)(Rax>(Ey)Sy)
   |(x)(Rax>(Ey)Sy)
   (x)(Rax>(Ey)Sy)
```

instance of it. The rule here is that the constant should have occurred before. But are we getting (Ey)Sy from Sa, or from Sb, or Sc, or Sd? Before we decide this, let's look at line 2. It is a universally quantified formula, and from it we can get, for example, (-Pa>(y)(Ray>Sa)), or (-Pb>(y)(Ray>Sb)), or,. . . Wait a second. Hold the phone. Line 6 contains the constant "b" after the predicate "P." So we now know we need to instantiate on line 2 with "b." Let's do it.

The remainder of the problem should now be a piece of cake. We are going to use DeM on line 6, and then pull -Pb out of that. Then we'll use MP on -Pb and line 8. This will give us (y)(Ray>Sb). We can then use UI on this to get (Rad>Sb). That will yield Sb, etc.

If you thought proofs were difficult before, you had no idea! But with a bit of practice you should become proficient at them.

```
1  -(x)(y)(PxvQy)                                     Premise
2  (x)(-Px>(y)(Ray>Sx))   /  (x)(Rax>(Ey)Sy)           Premise
3  (Ex)-(y)(PxvQy)                                    1, QEQ
4  (Ex)(Ey)-(PxvQy)                                   3, QEQ
5  |(Ey)-(PbvQy)                                      Ass
6  ||-(PbvQc)                                         Ass
7  |||Rad                                             Ass
8  |||(-Pb>(y)(Ray>Sb))                               2, UI
   |||     :
   |||     :
   |||(Ey)Sy
   ||(Rad>(Ey)Sy)
   ||(x)(Rax>(Ey)Sy)
   |(x)(Rax>(Ey)Sy)
   (x)(Rax>(Ey)Sy)
```

EXERCISES

A. (x)(Px>(y)Qxy)
(x)(y)(Qxy>Ryx)
/ (x)(Px>(Ey)Ryx)

B. (Ex)(y)(Pxy>Qxy)
(x)(y)(Qxy>Rxy)
/ ((x)Pxx>(Ey)Ryy)

C. -(x)(y)Pxy
(x)(y)(Pxy=Qxy)
/ (Ex)(Ey)-Qxy

D. (x)(y)(Qxy>Rxy)
-(x)Pxx / (Ex)(Ey)Qxy

E. (Ex)(Ey)Pxy
((x)(Ey)Pyx>(z)Qz
/ (x)(Rx>Qx)

F. (Ex)(y)-(Pxy.Qyx)
(x)(Pxx>Qxx)
((x)Pxxv(y)Ryy) / (Ey)Ray